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Annals of Pure and Applied Logic 127 (2004) 229–242

ANNALS OF PURE AND APPLIED LOGIC

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Between Galois connections and (some metamathematical) solutions of equations $fgf=f$ and $gfg=g$

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Abstract

The method based on the idea of Galois connection is well known. It facilitates investigations into similarities between mathematical structures, including isomorphisms between these structures, the highest degree of similarity. This idea is employed here and adapted so as to get to the core of aspects of the relationship between some metamathematical structures. The focus is put on the relation between traditional methodological orthodoxy based on the idea of proof (polished up with the help of the concept of closure operator), on the one hand, and on some alternative methodological set-ups based on other ideas such as consistency or some forms of maximality, on the other hand.

MSC: primary 06A15; secondary 28E10

Keywords: Galois (or antitone) connection; Isotone connection

1. Part I

Several solutions of equations $fgf=f$ and $gfg=g$, announced in Surma [4–7], as per the references at the end of this work, are a departure point of the present paper as well as a justification of its title.

Let $(P,\leqslant)$ be a *poset*, i.e. a set P with a binary relation $\leqslant$ of partial ordering. A mapping f from $(P,\leqslant)$ to $(Q,\leqslant)$ is called *isotone* iff $x\leqslant y$ implies that $fx\leqslant fy$, and it is *antitone* iff $x\leqslant y$ implies that $fy\leqslant fx$, for any $x,y\in P$. Given a mapping f from $(P,\leqslant)$ to $(Q,\leqslant)$ and a mapping g from $(Q,\leqslant)$ to $(P,\leqslant)$, we say that the pair (f,g)

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0168-0072/$ - see front matter
doi:10.1016/j.apal.2003.11.017

is isotone iff f and g are both isotone. Similarly, we call the pair (f,g) antitone just in case f and g are both antitone. Finally, we say that the pair (f,g) is a *weak connection* between $(P,\leqslant)$ and $(Q,\leqslant)$ iff both $x\leqslant gfx$ and $y\leqslant fgy$, for any $x\in P$ and $y\in Q$. In this context, we refer to f and g as *weak connectors* of the respective posets. Clearly, some weak connections are isotone and some antitone. The antitone weak connections are what we know from algebra under the name of Galois connections. We start by a brief summary discussion of the antitone connections. The following simple but useful fact is well known [1,2].

(1) If (f,g) is an antitone weak connection between $(P,\leqslant)$ and $(Q,\leqslant)$ then $fgfx=fx$ and $gfgy=gy$ for any $x\in P$ and $y\in Q$.

Also known is the following fact due to Schmidt [3].

(2) The following two conditions are equivalent: (i) (f,g) is an antitone weak connection between $(P,\leqslant)$ and $(Q,\leqslant)$; (ii) f is a mapping from $(P,\leqslant)$ to $(Q,\leqslant)$ and g a mapping from $(Q,\leqslant)$ to $(P,\leqslant)$ such that $x\leqslant gy$ is equivalent to $y\leqslant fx$ for any $x\in P$ and $y\in Q$.

None of the inequalities $x\leqslant gfx$ and $y\leqslant fgy$, appearing in the definition of a weak connection (f,g), can be strengthened to an equality simply on the basis of the definition of antitone weak connectionship. Yet, practical considerations justify the separation of this subclass from all antitone weak connections. We say that (f,g) is a strong connection between $(P,\leqslant)$ and $(Q,\leqslant)$, to be called from now on simply a connection between $(P,\leqslant)$ and $(Q,\leqslant)$, iff $gfx=x$ and $fgy=y$ for any $x\in P$ and $y\in Q$. Here f and g are referred to as *strong connectors* or simply as connectors of their respective posets. In some contexts involving connection (f,g), it may be useful to refer to the f as the *left connector* and treat the g as the *right connector*. Trivially, each antitone connection is an antitone weak connection. Furthermore, we have the following two facts involving these concepts (they are implicit in [1]).

(3) If (f,g) is an antitone weak connection between $(P,\leqslant)$ and $(Q,\leqslant)$ then the following three conditions are pairwise equivalent: (i) f is a connector of (f,g); (ii) g is onto; (iii) f is one-to-one.

(4) If (f,g) is an antitone connection between $(P,\leqslant)$ and $(Q,\leqslant)$ then f is one-to-one and g is a reverse of f.

Not all connections are antitone. Some connections of interest we encounter in practice are, in fact, isotone. This can justify a slight extension and/or modification of the original methodology based on the idea of Galois connection.

As already seen earlier, the definition of an antitone weak connection (f,g) is straightforward. Namely, (f,g) is an antitone weak connection just in case it is a weak connection as well as an antitone pair of mappings, and this combination already guarantees the deducibility of the equalities $fgfx=fx$ and $gfgy=gy$. The definition of an isotone weak connection is, however, less straightforward. Namely, we call (f,g) an isotone weak connection just in case the following three conditions are satisfied: (i) (f,g) is an isotone pair of mappings; (ii) (f,g) is a weak connection between their respective posets; and also (iii) (f,g) satisfies the equalities $fgfx=fx$ and $gfgy=gy$. The point here is that the latter condition (iii) cannot be deduced from conditions (i) and (ii) alone. The three facts below, which parallel facts (2)–(4), involve isotone connections.

(5) The conditions below are equivalent: (i) (f,g) is an isotone weak connection between $(P,\leqslant)$ and $(Q,\leqslant)$; (ii) f is a mapping from $(P,\leqslant)$ to $(Q,\leqslant)$ and g is a mapping from $(Q,\leqslant)$ to $(P,\leqslant)$ such that both, $x\leqslant gy$ is equivalent to $fx\leqslant y$, and $y\leqslant fx$ is equivalent to $gy\leqslant x$.

(6) If (f,g) is an isotone weak connection between $(P,\leqslant)$ and $(Q,\leqslant)$ then the following three conditions are pairwise equivalent: (i) f is a connector of (f,g); (ii) g is onto; (iii) f is one-to-one.

(7) If (f,g) is an isotone connection between $(P,\leqslant)$ and $(Q,\leqslant)$ then f is one-to-one and g is a reverse of f.

Clearly, fact (5), which involves isotone weak connections, is a counterpart of fact (2) (or the Schmidt Theorem) which involves antitone weak connections. As for facts (6) and (7), they are counterpart, respectively, of facts (3) and (4).

2. Part II

In this part, we deal with antitone as well as isotone connections in application to a few pairs of metamathematical structures. For convenience rather than for substance we shift, from now on, to the usual set-theoretic terminology and symbolic notation. In particular, symbols "$\subseteq$", "$\cup$" and "$\cap$" stand for the relation of set-inclusion, the operation of set-union and that of set-intersection, respectively. We denote by S the set of all sentences of an arbitrary but fixed object-language, and we use letters X, Y, $Z,\dots$ and A, B, $C,\dots$, with or without indices, to denote subsets of S and members of S, respectively. In this part of the paper, we only assume of S that it is a non-empty set of sentences of unspecified structure. The usual symbolic expression 2^Z denotes, of course, the powerset of Z, i.e. the set of all subsets of Z. We also write 2_Z to denote the class of all subsets of S which extend (or are supersets of) Z. The symbol $\emptyset$ stands for the empty set. For brevity, expressions of the kind of $\phi(X\cup\{A_1,A_2,\dots,A_n\})$ are being rendered throughout the rest of the paper simply as $\phi(X,A_1,A_2,\dots,A_n)$. The case of (Cons[Cn], Cn[Cons]). We define Cn as a *strongly regular closure operator* over S just in case it satisfies the following four conditions: (i) Cn is reflexive, i.e. $X\subseteq \mathrm{Cn}(X)$; (ii) Cn is monotonic, i.e. $X\subseteq Y$ implies that $\mathrm{Cn}(X)\subseteq \mathrm{Cn}(Y)$; (iii) Cn is idempotent, i.e. $\mathrm{Cn}(\mathrm{Cn}(X))\subseteq \mathrm{Cn}(X)$; and (iv) Cn is strongly regular, i.e. if $A\notin \mathrm{Cn}(X)$ then there is a set $Z\in 2_X$ such that $A\notin \mathrm{Cn}(Z)$ and $\mathrm{Cn}(Z,B)=S$ for any $B\notin Z$. Definitions (i)–(iii) are due to Tarski [8–10]. In this context, we also say that X is Cn-*maximal* just in case there is $A\notin \mathrm{Cn}(X)$ such that $A\in \mathrm{Cn}(X,B)$ for any $B\notin X$.

In a similar fasion, we define Cons as a *regular consistency* property over S just in case it satisfies all the conditions below: (i) Cons is non-trivial, i.e. $S\notin \mathrm{Cons}$; (ii) Cons is hereditary, i.e. $X\in \mathrm{Cons}\cap 2_Y$ implies that $Y\in \mathrm{Cons}$; and (iii) Cons is regular, i.e. $X\in \mathrm{Cons}$ implies that there is $Z\in \mathrm{Cons}\cap 2_X$ such that $Z'=Z$ for any $Z'\in \mathrm{Cons}\cap 2_Z$. Finally, we say that X is *Cons-maximal* iff $X\in \mathrm{Cons}$ and $Y=X$ for any $Y\in \mathrm{Cons}\cap 2_X$ (cf. [4]).

Pivotal to this case are the definition of Cons[Cn] in terms of a strongly regular closure operator Cn, i.e. the definition

$$X \in \mathrm{Cons[Cn]} \text{ iff } \mathrm{Cn}(X) \neq S$$

and the definition of Cn[Cons] in terms of a regular consistency property Cons, i.e.

$$A \in \mathrm{Cn[Cons]}(X) \text{ iff } A \in Y \text{ for any Cons-maximal extension } Y \text{ of } X.$$

These definitions, it will be seen, describe two important mappings. The first mapping, denoted conveniently as Cons[Cn], sends each strongly regular closure operator Cn over S to a regular consistency property over the same S. The second mapping, denoted as Cn[Cons], sends each regular consistency property Cons to a strongly regular closure operator. This is so because we have two lemmas as below.

Lemma 1. *If Cn is a strongly regular closure operator then* (i) Cons[Cn] *is a regular consistency property and* (ii) $\mathrm{Cn[Cons[Cn]]} = \mathrm{Cn}$.

Proof. (i) *Proof that* Cons[Cn] *is non-trivial*: By the hypothesis Cn is reflexive so $\mathrm{Cn}(S) = S$ and so $S \notin \mathrm{Cons[Cn]}$. This means that Cons[Cn] is non-trivial.

Proof that Cons[Cn] *is hereditary*: If $X \in \mathrm{Cons[Cn]} \cap 2_Y$ then $\mathrm{Cn}(X) \neq S$ and then $\mathrm{Cn}(Y) \neq S$ so $Y \in \mathrm{Cons[Cn]}$.

Proof that Cons[Cn] *is regular*: Suppose that $X \in \mathrm{Cons[Cn]}$, i.e. that $A \notin \mathrm{Cn}(X)$ for some A. By the hypothesis, Cn is strongly regular so there is Y such that (1) $Y \in 2_X$ and (2) $A \notin \mathrm{Cn}(Y)$ and $\mathrm{Cn}(Y,B) = S$ for any $B \notin Y$. From (2) $\mathrm{Cn}(Y) \neq S$. Hence by (1) we have (3) $Y \in \mathrm{Cons[Cn]} \cap 2_X$. Clearly, step (2) implies that (4) $Z = Y$ for any $Z \in \mathrm{Cons[Cn]} \cap 2_Y$. By (3) and (4) Cons[Cn] is regular.

(ii) *Proof that* $\mathrm{Cn[Cons[Cn]]} \subseteq \mathrm{Cn}$: Suppose that (1) $A \in \mathrm{Cn[Cons[Cn]]}(X)$ and (2) $A \notin \mathrm{Cn}(X)$. Cn is strongly regular. Hence, by (2) there is Y such that (3) $Y \in 2_X$ and (4) $A \notin \mathrm{Cn}(Y)$ and $\mathrm{Cn}(Y,B) = S$ for any $B \notin Y$. From (4) it follows that (5) $\mathrm{Cn}(Y) \neq S$. From (3) and (5) and the definition of Cons[Cn] we have step (6) $Y \in \mathrm{Cons[Cn]} \cap 2_X$. By (4) we also have step (7) $Z = Y$ for any Z such that $Z \in \mathrm{Cons[Cn]} \cap 2_Y$. From (6) and (7) we infer (8) Y is Cons[Cn]-maximal. From (1), (6), (8) and the definition of Cn[Cons[Cn]] in terms of Cons[Cn] we get (9) $A \in Y$. Cn is reflexive so we can conclude from (9) that $A \in \mathrm{Cn}(Y)$, contrary to (4).

Proof that $\mathrm{Cn} \subseteq \mathrm{Cn[Cons[Cn]]}$: Suppose that (1) $A \in \mathrm{Cn}(X)$ and (2) $A \notin \mathrm{Cn}[\mathrm{Cons[Cn]}](X)$. From (2) and the definition of Cn[Cons[Cn]] in terms of Cons[Cn] it follows that there is Y such that (3) $Y \in 2_X$, (4) Y is Cons[Cn]-maximal and (5) $A \notin Y$. From (4) it follows that (6) $Y \in \mathrm{Cons[Cn]}$ and (7) $Z = Y$ for any $Z \in \mathrm{Cons[Cn]} \cap 2_Y$. Cn is monotonic. Hence by (1) we have (8) $A \in \mathrm{Cn}(Y)$. Cn is a closure operator. Hence by (8) we get (9) $\mathrm{Cn}(Y,A) \subseteq \mathrm{Cn}(Y)$. From (6) and the definition of Cons[Cn] we draw (10) $\mathrm{Cn}(Y) \neq S$. Hence by (9) we have (11) $\mathrm{Cn}(Y,A) \neq S$. From (11) and the definition of Cons[Cn] we conclude that (12) $Y \cup \{A\} \in \mathrm{Cons[Cn]}$. It follows from (7) and (12) that $A \in Y$, contrary to (5). □

Lemma 2. *If* Cons *is a regular consistency property then*: (i) Cn[Cons] *is a strongly regular closure operator*; *and* (ii) $\mathrm{Cons[Cn[Cons]]} = \mathrm{Cons}$.

Proof. (i) *Proof that* Cn[Cons] *is reflexive*: If $A \notin \mathrm{Cn[Cons]}(X)$ then by the definition of Cn[Cons] there is a Cons-maximal Y such that $Y \in 2_X$ and $A \notin Y$ so $A \notin X$.

Proof that Cn[Cons] *is monotonic*: Suppose that (1) $X \subseteq Y$, (2) $A \in \mathrm{Cn[Cons]}(X)$ and (3) $A \notin \mathrm{Cn[Cons]}(Y)$. By (3) and the definition of Cn[Cons] there is Z such that (4) Z is Cons-maximal, (5) $Z \in 2_Y$ and (6) $A \notin Z$. From (1) and (5) it follows that (7) $Z \in 2_X$. From (2), (4), (7) and the definition of Cn[Cons] we have $A \in Z$, contrary to (6).

Proof that Cn[Cons] *is idempotent*: Suppose that (1) $A \notin \mathrm{Cn[Cons]}(X)$. From (1) and the definition of Cn[Cons] there is Y such that (2) $Y \in 2_X$, (3) Y is Cons-maximal and (4) $A \notin Y$. From (3) and the definition of a Cons-maximal set we infer that (5) $Y \in \mathrm{Cons}$ and $Z = Y$ for any $Z \in \mathrm{Cons} \cap 2_Y$. As already proved, Cn[Cons] is monotonic. Hence by (2) we have (6) $\mathrm{Cn[Cons]}(X) \subseteq \mathrm{Cn[Cons]}(Y)$. We prove now (7) $\mathrm{Cn[Cons]}(Y) \subseteq Y$. Indeed, if $B \in \mathrm{Cn[Cons]}(X)$ then, by the definition of Cn[Cons], $B \in Y$ is implied by the fact that Y is Cons-maximal. Hence, by (3), $B \in Y$. Using (6) and (7) we infer that (8) $\mathrm{Cn[Cons]}(X) \subseteq Y$. By the definition of Cn[Cons](Cn[Cons]) in terms of Cn[Cons], it follows from (3), (4) and (8) that (9) $A \notin \mathrm{Cn[Cons(Cn[Cons]}(X))$.

Proof that Cn[Cons] *is strongly regular*: If $A \notin \mathrm{Cn[Cons]}(X)$ then by the definition of Cn[Cons] there is Y such that (1) Y is Cons-maximal, (2) $Y \in 2_X$ and (3) $A \notin Y$. By (1) we have (4) $Y \in \mathrm{Cons}$ and (5) $Z = Y$ for any $Z \in \mathrm{Cons} \cap 2_Y$. To conclude that Cn[Cons] is strongly regular we need steps (6) $A \notin \mathrm{Cn[Cons]}(Y)$ and (7) $\mathrm{Cn[Cons]}(Y, B) = S$ for any $B \notin Y$. Step (6) follows from (2), (3) and (5) while step (7) follows from (5).

(ii) *Proof that* $\mathrm{Cons[Cn[Cons]]} \subseteq \mathrm{Cons}$: Suppose that (1) $X \in \mathrm{Cons[Cn[Cons]]}$. By the hypothesis Cons is a regular consistency property. Hence, by Lemma 2(i), Cn[Cons] is a strongly regular closure operator and, by Lemma 1(i), Cons[Cn[Cons]] is a regular consistency property. By (1) and the definition of Cons[Cn[Cons]] in terms of Cn[Cons] we conclude that (2) $\mathrm{Cn[Cons]}(X) \neq S$. By (2) there is $A \in S$ such that (3) $A \notin \mathrm{Cn[Cons]}(X)$. By (3) and the definition of Cn[Cons] in terms of Cons there is Y such that (4) $Y \in 2_X$, (5) Y is Cons-maximal and (6) $A \notin Y$. It follows from (5) that (7) $Y \in \mathrm{Cons}$ and (8) $Z = Y$ for any Z such that $Z \in \mathrm{Cons} \cap 2_Y$. Cons is hereditary. Hence by (4) and (7) we get (9) $X \in \mathrm{Cons}$.

Proof that $\mathrm{Cons} \subseteq \mathrm{Cons[Cn[Cons]]}$: Suppose that (1) $X \in \mathrm{Cons}$ and (2) $X \notin$ Cons[Cn [Cons]]. Cons is a regular consistency property. Hence by (1) there is Y such that (3) $Y \in \mathrm{Cons}$, (4) $Y \in 2_X$ and (5) $Z = Y$ for any Z such that $Z \in \mathrm{Cons} \cap 2_Y$. From (3) and (5) we have (6) Y is Cons-maximal. From (2) and the definition of Cons[Cn[Cons]] in terms of Cn[Cons] we have (7) $\mathrm{Cn[Cons]}(X) = S$. Steps (4) and (6) imply that (8) $\mathrm{Cn[Cons]}(X) \subseteq Y$. From (7) and (8) we conclude that (9) $Y = S$. Using (3) and (9) we conclude that $S \in \mathrm{Cons}$, i.e. that Cons is trivial, contrary to the hypothesis. □

Lemmas 1 and 2 imply the following conclusion.

Corollary 1. *The pair of mappings* (Cons[Cn], Cn[Cons]) *is a connection between strongly regular closure operators and regular consistency properties over one and the same set S.*

The cases of Lindenbaum operators Ln and of the families of maximal sets Max and their interaction with Cn and Cons. To describe more connections similar to that of Corollary 1, we introduce the following terminology (Cf. [6,7]). We say that Ln is a *Lindenbaum operator* just in case it satisfies all the five conditions below. (i) Ln is non-trivial, i.e. $\mathrm{Ln}(S)=\emptyset$; (ii) Ln is extensive, i.e. $\mathrm{Ln}(X)\subseteq 2_X$; (iii) Ln is inclusive, i.e. $\mathrm{Ln}(\emptyset)\cap 2_X\subseteq \mathrm{Ln}(X)$; (iv) Ln is antimonotonic, i.e. $X\subseteq Y$ implies that $\mathrm{Ln}(Y)\subseteq \mathrm{Ln}(X)$, and (v) Ln is regular, i.e. the fact that $X\in\mathrm{Ln}(\emptyset)$ and $Y\in\mathrm{Ln}(X)$ implies that $X=Y$.

Furthermore, we define Max to be a *family of maximal sets* just in case it satisfies the following two conditions: (i) Max is non-trivial, i.e. $S\notin\mathrm{Max}$, and (ii) Max is regular, i.e. $X=Y$ for any $X\in\mathrm{Max}$ and any $Y\in\mathrm{Max}\cap 2_X$.

Finally, we accept the following pivotal pairs of definitions:

(i) $X\in\mathrm{Ln}[\mathrm{Cn}](Y)$ iff $X\in 2_Y$, $\mathrm{Cn}(X)\neq S$, and $Z=X$ for any $Z\in 2_X$ such that $\mathrm{Cn}(Z)\neq S$. $A\in\mathrm{Cn}[\mathrm{Ln}](X)$ iff $\mathrm{Ln}(X)\subseteq\mathrm{Ln}(X,A)$.

(ii) $A\in\mathrm{Cn}[\mathrm{Max}](X)$ iff $A\in Y$ for any $Y\in\mathrm{Max}\cap 2_X$. $X\in\mathrm{Max}[\mathrm{Cn}]$ iff $\mathrm{Cn}(X)\neq S$, and $\mathrm{Cn}(X,B)=S$ for any $B\notin X$.

(iii) $X\in\mathrm{Cons}[\mathrm{Ln}]$ iff $\mathrm{Ln}(X)\neq\emptyset$. $X\in\mathrm{Ln}[\mathrm{Cons}](Y)$ iff $X\in\mathrm{Cons}\cap 2_Y$ and $Z=X$ for any $Z\in\mathrm{Cons}\cap 2_X$.

(iv) $X\in\mathrm{Cons}[\mathrm{Max}]$ iff $\mathrm{Max}\cap 2_X\neq\emptyset$. $X\in\mathrm{Max}[\mathrm{Cons}]$ iff $X\in\mathrm{Cons}$ and $Y=X$ for any $Y\in\mathrm{Cons}\cap 2_X$.

(v) $X\in\mathrm{Max}[\mathrm{Ln}]$ iff $X\in\mathrm{Ln}(\emptyset)$. $X\in\mathrm{Ln}[\mathrm{Max}](Y)$ iff $X\in\mathrm{Max}\cap 2_Y$.

Like in the case of (Cons[Cn], Cn[Cons]) each of these pairs of definitions describes a connection between their respective posets. By an argument similar to that leading to our Corollary 1, we arrive at the following statement.

Corollary 2. (i) $(\mathrm{Ln}[\mathrm{Cn}],\mathrm{Cn}[\mathrm{Ln}])$ *is a connection between strongly regular closure operators and Lindenbaum operators over* S.

(ii) $(\mathrm{Cn}[\mathrm{Max}],\mathrm{Max}[\mathrm{Cn}])$ *is a connection between families of maximal sets and strongly regular closure operators over* S.

(iii) $(\mathrm{Cons}[\mathrm{Ln}],\mathrm{Ln}[\mathrm{Cons}])$ *is a connection between Lindenbaum operators and regular consistency properties over* S.

(iv) $(\mathrm{Cons}[\mathrm{Max}],\mathrm{Max}[\mathrm{Cons}])$ *is a connection between families of maximal sets and regular consistency properties over* S

(v) $(\mathrm{Max}[\mathrm{Ln}],\mathrm{Ln}[\mathrm{Max}])$ *is a connection between Lindenbaum operators and families of maximal sets over* S.

It is of some interest that statement 2(v), i.e. the last part of Corollary 2, can be improved. Namely, we have the following theorem.

Theorem 1. $(\mathrm{Max}[\mathrm{Ln}],\mathrm{Ln}[\mathrm{Max}])$ *is an isotone connection between Lindenbaum operators and families of maximal sets over* S.

Proof. By Corollary 2(v) we only need to show that (Max[Ln], Ln[Max]) is isotone. *Proof that* Max[Ln] *is isotone*: Suppose that (1) $\mathrm{Ln}_1\subseteq\mathrm{Ln}_2$. Clearly, if $X\in\mathrm{Max}[\mathrm{Ln}_1]$ then, by the definition of $\mathrm{Max}[\mathrm{Ln}_1]$, we get that $X\in\mathrm{Ln}_1(\emptyset)$ then, by (1), $X\in\mathrm{Ln}_2(\emptyset)$

and then, by the definition of $\mathrm{Max}[\mathrm{Ln}_2]$, $X \in \mathrm{Max}[\mathrm{Ln}_2]$. Proof that Ln[Max] is isotone is similar to that of the previous one. □

Unlike in Theorem 1, where (Max[Ln], Ln[Max]) is proven isotone, the answer to the question of whether any one of the remaining pairs (Cons[Cn], Cn[Cons]), (Ln[Cn], Cn[Ln]), (Cn[Max], Max[Cn]), (Cons[Ln], Ln[Cons]) and (Cons[Max], Max[Cons]) is either isotone or antitone is not warranted within the present conceptual set-up. Only partial answers, all summarised by the lemma below, are available under that conceptual set-up.

Lemma 3. (i) *If* Cn *is a strongly regular closure operator then* Cons[Cn] *and* Ln[Cn] *are antitone*. (ii) *If* Ln *is a Lindenbaum operator then* Cons[Ln] *is isotone*. (iii) *If* Max *is a family of maximal sets then* Cn[Max] *is antitone and* Cons[Max] *is isotone*.

Proof. Here we only provide proof that Ln[Cn] is antitone. Proofs of the remaining cases are omitted as straightforward. Suppose that (1) $\mathrm{Cn}_1 \subseteq \mathrm{Cn}_2$, (2) $X \in \mathrm{Ln}[\mathrm{Cn}_2](Y)$ and (3) $X \notin \mathrm{Ln}[\mathrm{Cn}_1](Y)$. By (3) we can state that (4) either $X \notin 2_Y$ or $\mathrm{Cn}_1(X) = S$ or $Z \in 2_X$ and $\mathrm{Cn}_1(Z) \neq S$ and $Z \neq X$ for some Z. By (2), on the other hand, we have (5) $X \in 2_Y$, (6) $\mathrm{Cn}_2(X) \neq S$ and (7) $Z = X$ for any $Z \in 2_X$ such that $\mathrm{Cn}_2(Z) \neq S$. By (1) and (6) we get step (8) $\mathrm{Cn}_1(X) \neq S$. By (4), (5) and (8) there is Z such that (9) $Z \in 2_X$, (10) $\mathrm{Cn}_1(Z) \neq S$ and (11) $Z \neq X$. It follows from (6), (7) and (9) that $Z = X$, contrary to (11). Thus Ln[Cn] is antitone. □

3. Part III

To improve on the partial results of Lemma 3 a more comprehensive conceptual set-up is needed. Indeed, until now the references to the set S (of sentences of a fixed object language) depended on no logical constants. From now on we assume that the object language has a fixed "inner" structure so that the elements of S can be identified either as simple (or basic) sentences or as compound sentences, i.e. sentences made up of simpler ones with the help of some specified logical constants. For the sake of brevity in the rest of this paper, we restrict ourselves to negation $\neg$ as the only such logical constant. This restriction, however, is not essential. An approach similar to that involving negation as a logical constant extends to any other standard logical constant (or constants). To save space, however, all the details involving logical constants other than negation $\neg$ are omitted.

We proceed by first quoting all the necessary terminology involving negation $\neg$. We say that X is ($\neg$)-*complete* iff $(\neg A) \in X$ is equivalent to the fact that $A \notin X$ for any A. Two particular cases of this concept prove useful in our context. Namely, X is top *down* ($\neg$)-*complete* iff $(\neg A) \in X$ implies that $A \notin X$. And, X is *bottom up* ($\neg$)-*complete* iff $(\neg A) \in X$ is implied by the fact that $A \notin X$. We use ($\neg$)-completeness in order to define the concept of ($\neg$)-saturation, the main concept of this part of the paper, to be applied to Cn, Cons, Ln and Max, respectively.

In application to Cn the definition of $(\neg)$-saturation runs as follows. Cn is called $(\neg)$-*saturated* iff X is $(\neg)$-complete for any Cn-maximal X. We say that Cn is *top down* $(\neg)$-*saturated* iff X is top down $(\neg)$-complete for any Cn-maximal X. And Cn is *bottom up* $(\neg)$-*saturated* iff X is bottom up $(\neg)$-complete for any Cn-maximal X.

The respective terminology for Cons, Ln and Max looks almost like a duplication of the above terminology for Cn. Namely, Cons is called $(\neg)$-saturated iff X is $(\neg)$-complete for any Cons-maximal X. And Cons is top down $(\neg)$-saturated (respectively, bottom up $(\neg)$-saturated) iff X is top down $(\neg)$-complete (respectively, bottom up $(\neg)$-complete) for any Cons-maximal X. Furthermore, Ln is $(\neg)$-saturated (respectively, top down $(\neg)$-saturated/bottom up $(\neg)$-saturated) iff X is $(\neg)$-complete (respectively, top down $(\neg)$-complete/bottom up $(\neg)$-complete) for any $X \in \mathrm{Ln}(\emptyset)$.

Finally, Max is $(\neg)$-saturated (respectively, top down/bottom up $(\neg)$-saturated) iff X is $(\neg)$-complete (respectively, top down/bottom up $(\neg)$-complete) for any $X \in \mathrm{Max}$.

Theorem 2. *Each of the pairs* (Cons[Cn], Cn[Cons]), (Ln[Cn], Cn[Ln]) *and* (Cn[Max], Max[Cn]) *is a Galois* (*or antitone*) *connection between their respective* $(\neg)$-*saturated posets over* S.

Proof. *Proof of the case of* (Cons[Cn], Cn[Cons]): By Corollary 2 this pair is a connection while by Lemma 3(i) its left connector Cons[Cn] is antitone. It follows that to complete the proof we only need to justify that Cons[Cn] is $(\neg)$-saturated provided that Cn is a strongly regular $(\neg)$-saturated closure operator, and that Cn[Cons] is both $(\neg)$-saturated and antitone provided that Cons is a $(\neg)$-saturated consistency property.

Proof that Cons[Cn] *is* $(\neg)$-*saturated*: Following the definition of a $(\neg)$-saturated consistency property we need to show that X is $(\neg)$-complete for any Cons[Cn]-maximal X. To this end suppose that X is Cons[Cn]-maximal, i.e. that (1) $X \in \mathrm{Cons[Cn]}$ and (2) $Y = X$ for any $Y \in \mathrm{Cons[Cn]} \cap 2_X$. By (1) and the definition of Cons[Cn] there is A such that (3) $A \notin \mathrm{Cn}(X)$. By the hypothesis Cn is strongly regular. Hence by (3) there is Z such that (4) $Z \in 2_X$, (5) $A \notin \mathrm{Cn}(Z)$ and (6) $\mathrm{Cn}(Z,B) = S$ for any $B \notin Z$. From 5 and 6 we infer that (7) Z is Cn-maximal. By the hypothesis, Cn is $(\neg)$-saturated. Hence by (7) we have (8) Z is $(\neg)$-complete. Now it follows from (5) that $\mathrm{Cn}(Z) \neq S$, i.e. that (9) $Z \in \mathrm{Cons[Cn]}$. From (2), (4) and (9) we get (10) $Z = X$. Finally, by (8) and (10) X is $(\neg)$-complete. This shows that Cons[Cn] is $(\neg)$-saturated.

Proof that Cn[Cons] *is* $(\neg)$-*saturated*: If X is Cn[Cons]-maximal then there is A such that (1) $A \notin \mathrm{Cn[Cons]}(X)$ and (2) $A \in \mathrm{Cn[Cons]}(X,B)$ for any $B \notin X$. By (1) and the definition of Cn[Cons] there is Y such that (3) $Y \in 2_X$, (4) Y is Cons-maximal and (5) $A \notin Y$. We claim now that (6) $Y = X$. Indeed, if $Y \neq X$ then, by (3), $B \in Y$ and $B \notin X$ for some B. It follows from (2) that $A \in \mathrm{Cn[Cons]}(X,B)$ and hence, by (3), $A \in \mathrm{Cn[Cons]}(Y)$. Using the definition of Cn[Cons], we conclude that $A \in Z$ for any Cons-maximal Z such that $Z \in 2_Y$. Finally, by (4), $A \in Y$ contrary to (5). This proves step (6). It follows from (4) and (6) that (7) X is Cons-maximal. Cons is $(\neg)$-saturated. Hence, by (7), X is $(\neg)$-complete.

Proof that Cn[Cons] *is antitone*: Suppose that (1) $\mathrm{Cons}_1 \subseteq \mathrm{Cons}_2$, (2) $A \in \mathrm{Cn[Cons_2]}(X)$ and (3) $A \notin \mathrm{Cn[Cons_1]}(X)$. By (3) and the definition of $\mathrm{Cn[Cons_1]}$ there is Y such that (4) Y is $\mathrm{Cons_1}$-maximal, (5) $Y \in 2_X$ and (6) $A \notin Y$. By (4) we conclude that (7)

$Y \in \mathrm{Cons}_1$ and that (8) $Z = Y$ for any Z such that $Z \in \mathrm{Cons}_1 \cap 2_Y$. By (8) we have that (9) $B \in Y$ for any B such that $Y \cup \{B\} \in \mathrm{Cons}_1$. On the other hand, by (2) and the definition of Cn[Cons$_2$] we infer that (10) the fact that Y is Cons$_2$-maximal and $Y \in 2_X$ implies that $A \in Y$. From (5), (6) and (10) it follows that (11) Y is not Cons$_2$-maximal. From (11) we get step (12) $Y \notin \mathrm{Cons}_2$ or there is $Z \in \mathrm{Cons}_2 \cap 2_Y$ such that $Z \neq Y$. By (1), (7) and (12) there is Z such that (13) $Z \in \mathrm{Cons}_2$, (14) $Z \in 2_Y$ and (15) $Z \neq Y$. By (14) and (15) there is B such that (16) $B \in Z$ and (17) $B \notin Y$. By the hypothesis Cons$_1$ is bottom up $(\neg)$-saturated. Hence by (4) and (17) we infer (18) $\neg B \in Y$. By (14), (16) and (18) we conclude that (19) B, $\neg B \in Z$. It follows from (4) and (19) that Cons$_1$ is not top down $(\neg)$-saturated contrary to the hypothesis. This completes the proof that (Cons[Cn], Cn[Cons]) is a Galois connection.

The proof that the remaining two pairs (Ln[Cn], Cn[Ln]) and (Cn[Max], Max[Cn]) are also Galois connections can be provided along the lines similar to that of the proof of the first case but to save space this proof is omitted here. $\square$

Finally, we state the following theorem.

Theorem 3. *Each of the pairs* (Cons[Ln], Ln[Cons]) *and* (Cons[Max], Max[Cons]) *is an isotone connection between their respective* $(\neg)$-*saturated posets over* S.

Proof. Here, we only prove that (Cons[Ln], Ln[Cons]) is an isotone connection. The proof of fact that (Cons[Max], Max[Cons]) is also an isotone connection is similar to that of the first claim.

By Corollary 2(iii) the pair (Cons[Ln], Ln[Cons]) is a connection and by Lemma 3(ii) its left connector Cons[Ln] is isotone. In this light, we only need to show that Cons[Ln] is $(\neg)$-saturated provided that Ln is a $(\neg)$-saturated Lindenbaum operator, and that Ln[Cons] is both $(\neg)$-saturated and isotone provided that Cons is a $(\neg)$-saturated consistency property.

Proof that Cons[Ln] *is* $(\neg)$-*saturated*: To this end if X is Cons[Ln]-maximal then (1) $X \in \mathrm{Cons[Ln]}$ and $Y = X$ for any $Y \in \mathrm{Cons[Ln]} \cap 2_X$. By (1) and the definition of Cons[Ln], $\mathrm{Ln}(X) \neq \emptyset$ so that $Z \in \mathrm{Ln}(X)$ for some Z. Ln is antimonotonic and extensive. Hence (2) $Z \in \mathrm{Ln}(\emptyset) \cap 2_X$. We now need to prove that (3) $Z = X$. Suppose, to the contrary, that $Z \neq X$, i.e. that $A \in Z$ and $A \notin X$ for some A. It follows that $X \cup \{A\} \neq X$ so, by (1), $X \cup \{A\} \notin \mathrm{Cons[Ln]}$. On the other hand, it follows from (2) that $X \cup \{A\} \subseteq Z$. Cons[Ln] is hereditary so $Z \notin \mathrm{Cons[Ln]}$ contrary to (1). This proves step (3). By (2) and (3) we have (4) $X \in \mathrm{Ln}(\emptyset)$. Ln is $(\neg)$-saturated so, by (4), X is $(\neg)$-complete.

Proof that Ln[Cons] *is* $(\neg)$-*saturated*: If $X \in \mathrm{Ln[Cons]}(\emptyset)$ then, by the definition of Ln[Cons], $X \in \mathrm{Cons}$ and $Y = X$ for any $Y \in \mathrm{Cons} \cap 2_X$ and then X is Cons-maximal. It follows that Ln[Cons] is $(\neg)$-saturated because, by the hypothesis, Cons is $(\neg)$-saturated.

Proof that Ln[Cons] *is isotone*: Suppose that (1) $\mathrm{Cons}_1 \subseteq \mathrm{Cons}_2$, (2) $X \in \mathrm{Ln[Cons_1]}(Y)$ and (3) $X \notin \mathrm{Ln[Cons_2]}(Y)$. By (3) and the definition of Ln[Cons$_2$] we infer that (4) $X \notin \mathrm{Cons}_2 \cap 2_Y$ or $Z \neq X$ for some $Z \in \mathrm{Cons}_2 \cap 2_X$. By (2) and the definition of Ln[Cons$_1$] we have steps (5) $X \in \mathrm{Cons}_1 \cap 2_Y$ and (6) $Z = X$ for any $Z \in \mathrm{Cons}_2 \cap 2_X$. By (1), (4) and (5) there is Z such that (7) $Z \in \mathrm{Cons}_2 \cap 2_X$ and (8) $Z \neq X$. By (7) and (8) there is A such that (9) $A \in Z$ and (10) $A \notin X$. Steps (5) and (6) imply that (11) X is Cons_2-maximal. By the hypothesis, Cons_1 is $(\neg)$-saturated. Hence by (10) and (11) it follows that (12) $\neg A \in X$. Cons_2 is regular. Hence by (7) there is U such that (13) $U \in \text{Cons}_1 \cap 2_Z$ and (14) $U^* = U$ for any $U^* \in \text{Cons}_2 \cap 2_U$. From (13) and (14) we infer that (15) X is Cons_2-maximal. And from (7) and (13) we get step (16) $U \in 2_X$. Steps (12) and (16) imply that (17) $\neg A \in U$. On the other hand, from (15) and (17) we conclude that (18) $A \notin U$ because Cons_2 is also $(\neg)$-saturated. It follows from (13) and (18) that $A \notin Z$ contrary to (9). $\square$

4. Part IV

(Cons[Cn], Cn[Cons]) re-visited. Among the connections, either isotone or antitone, investigated into and summarised as Theorems 1–3 in Parts II and III, the case of (Cons[Cn], Cn[Cons]) seems to prompt further interest.

To recall, connector Cons[Cn] of the Galois connection (Cons[Cn], Cn[Cons]) acts on the set of all strongly regular $(\neg)$-saturated closure operators Cn over S. And an inspection of the proof-process leading to our Theorems 2 and 3 shows that it is the strong regularity of Cn which is essential to its definition. The same holds true of Cn[Cons], i.e. of the other connector, acting on the set of all regular $(\neg)$-saturated consistency properties Cons, where the regularity of Cons plays an equally essential role in its definition.

Strong regularity of Cn and regularity of Cons deserve mention for their contribution to the resulting simplicity and transparency of our proofs of Theorems 2 and 3. The same strong regularity of Cn and regularity of Cons also helped in making our pivotal definitions of Cons[Cn] and Cn[Cons] independent of negation $\neg$ or of any other logical constants for that matter. The references to negation $\neg$ in Part III, however, were necessary there in order to identify some members of S as negated sentences, something that facilitated the proof that Cn[Cons] is antitone.

At the same time, strong regularity of Cn and regularity of Cons became an impediment to further extensions of the reach and scope of the Galois connection under discussion. They exclude from consideration all strongly irregular closure operators and irregular consistency properties. Also they do not help in discerning finitary closure operators from infinitary ones or compact consistency properties from incompact ones.

In what follows we focuss on this option by providing definitions of what we call finitary $(\neg)$-closure operators and compact $(\neg)$-consistency properties and showing that these two structures are Galois-connected. In some of the definitions to follow we use "$X \in \text{fin}(Y)$", an extra piece of our symbolic notation, to stand for the fact that X is a finite subset of Y. Also to avoid confusion with the notation used in the earlier parts of this work we will use from now on symbols "cn" and "cons" rather the symbols "Cn" and "Cons" which were used in Parts II and III.

Our new definitions involving closure operators are as follows. We say that cn is *finitary* iff $\text{cn}(X) \subseteq \cup\{\text{cn}(Y)\colon Y \in \text{fin}(X)\}$ for any X. Next, cn is called $(\neg)$-*analytic* iff $\text{cn}(A, \neg A) = S$ for any A. Then we say that cn is $(\neg)$-*synthetic* iff $cn(X, A) \cap \text{cn}(X, \neg A)$

$\subseteq \mathrm{cn}(X)$ for any A and X. And, finally, cn is defined as a $(\neg)$-*closure operator* iff it is a closure operator which is both $(\neg)$-analytic and $(\neg)$-synthetic. It is a well-known fact, due to Lindenbaum and Tarski [8–10] that under these definitions the condition for a strongly regular cn is obsolete so we are given the following lemma.

Lemma 4. *Each finitary* $(\neg)$*-closure operator is strongly regular.*

Clearly, by this lemma, the strong regularity of a closure operator ceases to function as a primitive concepts in this part of the paper.

We also provide a similar set of definitions for the consistency properties. We begin by saying that cons is *compact* iff, for any X, the fact that $X \notin \mathrm{cons}$ implies that $Y \notin \mathrm{cons}$ for some $Y \in \mathrm{fin}(X)$. Next, we say that cons is $(\neg)$-*analytic* iff $\{A, \neg A\} \notin \mathrm{cons}$ for any A. And that cons is $(\neg)$-*synthetic* iff the fact that $X \in \mathrm{cons}$ implies that $X \cup \{A\} \in \mathrm{cons}$ or $X \cup \{\neg A\} \in \mathrm{cons}$ for any A and X. Finally, we say that cons is a $(\neg)$-*consistency property* iff cons is a consistency property which is both $(\neg)$-analytic and $(\neg)$-synthetic. Given these definitions we can now state the following lemma whose proof is similar to that of Lemma 4.

Lemma 5. *Each compact consistency property is regular.*

This lemma, too, makes the concept of a regular consistency property as a primitive concept obsolete. We adopt the following pair of pivotal definitions. The definition of cons[cn] in terms of a finitary $(\neg)$-closure operator cn, i.e.

$$X \in \mathrm{cons}[\mathrm{cn}] \text{ iff } \mathrm{cn}(X) \neq S$$

and the definition of cn[cons] in terms of a compact $(\neg)$-consistency property, i.e.

$$A \in \mathrm{cn}[\mathrm{cons}](X) \text{ iff } X \cup \{\neg A\} \notin \mathrm{cons}.$$

The present definition of cons[cn] and the definition of Cons[Cn], as used in Parts II and III, may look, structurally, as one and the same definition. But they are different because what they apply to, i.e. cn and Cn are defined as different objects. As for our second pivotal definition of cn[cons], it is different from that of Cn[Cons] not only structurally but also because it makes the concept of cn[cons] dependent not only on cons but also on negation $\neg$. Based on these new pivotal definitions we proceed now to the stating and proving of three lemmas as follows.

Lemma 6. *If cn is a finitary* $(\neg)$*-closure operator then* (i) cons[cn] *is a compact* $(\neg)$*-consistency property and* (ii) $\mathrm{cn}[\mathrm{cons}[\mathrm{cn}]] = \mathrm{cn}$.

Proof. (i) Proof that cons[cn] is non-trivial and hereditary is implied directly by the fact that cn is reflexive and monotonic.

Proof that cons[cn] *is compact*: Indeed, if $X \notin \mathrm{cons}[\mathrm{cn}]$ then, by the definition of cons[cn], $\mathrm{cn}(X) = S$ and hence $A \in \mathrm{cn}(X)$ for any A. By the hypothesis, cn is finitary so there is $Y \in \mathrm{fin}(X)$ such that $\mathrm{cn}(Y) = S$. Then, by another use of the definition of cons[cn] there is then $Y \in \mathrm{fin}(X)$ such that $Y \notin$ cons[cn]. This means that cons[cn] is compact.

Proof that cons[cn] *is* $(\neg)$-*analytic*: If cons[cn] were not $(\neg)$-analytic, i.e. $\{A, \neg A\} \in$ cons[cn] then we would have that $\mathrm{cn}(A, \neg A) \neq S$, i.e. that cn were not $(\neg)$-analytic contrary to our hypothesis.

Proof that cons[cn] *is* $(\neg)$-*synthetic*: If cons[cn] were not $(\neg)$-synthetic, i.e. if $X \in$ cons[cn] while $X \cup \{A\} \notin$ cons[cn] and $X \cup \{\neg A\} \notin$ cons[cn] for some A and X then $\mathrm{cn}(X) \neq S$, $\mathrm{cn}(X, A) = S$ and $\mathrm{cn}(X, \neg A) = S$, i.e. cn, too, were not $(\neg)$-synthetic contrary to the hypothesis.

(ii) *Proof that* cn[cons[cn]] $\subseteq$ cn: Suppose that (1) $A \in$ cn[cons[cn]](X) and $A \notin$ cn (X). By (1) and the definition of cn[cons[cn]] in terms of cons[cn] $X \cup \{\neg A\} \notin$ cons[cn] and hence (2) $\mathrm{cn}(X, \neg A) = S$. By the hypothesis, cn is $(\neg)$-synthetic so it follows from (1) that (3) $A \notin \mathrm{cn}(X, A) \cap \mathrm{cn}(X, \neg A)$. But cn is reflexive. Hence, by (3), $A \notin \mathrm{cn}(X, \neg A)$ so $\mathrm{cn}(X, \neg A) \neq S$ contrary to (2). This proves the inclusion in question.

Proof that cn $\subseteq$ cn[cons[cn]]: If $A \in \mathrm{cn}(X)$ then $\mathrm{cn}(X, \neg A) = S$ because cn is $(\neg)$-analytic. Hence, by the definition of cons[cn], $X \cup \{\neg A\} \notin$ cons[cn] and hence, by the definition of cn[cons[cn]] in terms of cons[cn], $A \in$ cn[cons[cn]](X). $\square$

Lemma 7. *If* cons *is a compact* $(\neg)$-*consistency property then* (i) cn[cons] *is a finitary* $(\neg)$-*closure operator and* (ii) cons[cn[cons]] = cons.

Proof. (i) *Proof that* cn[cons] *is reflexive*: If $A \in X$ and $A \notin$ cn[cons](X) then $X \cup \{\neg A\} \in$ cons and then $X \cup \{A, \neg A\} \in$ cons which means that cons is not $(\neg)$-analytic contrary to our hypothesis.

Proof that cn[cons] *is monotonic*: If $Y \in 2_X$ and $A \in$ cn[cons](X) then, by the definition of cn[cons], $X \cup \{\neg A\} \notin$ cons and hence $Y \cup \{\neg A\} \notin$ cons because cons is hereditary. By another use of the definition of cn[cons] $A \in$ cn[cons](Y).

Proof that cn[cons] *is idempotent*: Here, we begin by proving that for any X (1) $X \in$ cons implies that cn[cons]$(X) \in$ cons. To this end let (1.1) $X \in$ cons and suppose, to the contrary, that (1.2) cn[cons]$(X) \notin$ cons. By the hypothesis cons is compact. Hence by (1.2) there is Y such that (1.3) $Y \in \mathrm{fin}(\mathrm{cn[cons]}(X))$ and (1.4) $Y \notin$ cons. By (1.3) there are n and $A_1, A_2, \ldots, A_n$ such that (1.5) $Y = \{A_1, A_2, \ldots, A_n\}$ and that (1.6) $A_1, A_2, \ldots, A_n \in$ cn[cons](X). Using the definition of cn[cons] we draw from (1.6) that (1.7) $X \cup \{\neg A_1\} \notin$ cons, $X \cup \{\neg A_2\} \notin$ cons$, \ldots, X \cup \{\neg A_n\} \notin$ cons. By (1.1) and (1.7) we conclude that (1.8) $X \cup \{A_1\} \in$ cons, $X \cup \{A_2\} \in$ cons$, \ldots, X \cup \{A_n\} \in$ cons because cons is $(\neg)$-synthetic. By the hypothesis cons is compact. Hence by (1.1) and Lemma 5 there is Z such that (1.9) $Z \in 2_X$, (1.10) $Z \in$ cons and (1.11) $U = Z$ for any $U \in \mathrm{cons} \cap 2_X$.

Steps (1.7), (1.9) and (1.10) imply that (1.12) $Z \cup \{A_1\} \in$ cons, $Z \cup \{A_2\} \in$ cons$, \ldots,$ $Z \cup \{A_n\} \in$ cons because cons is both hereditary and $(\neg)$-synthetic. From (1.11) and (1.12) we can infer step (1.13) $A_1, A_2, \ldots, A_n \in Z$. And from (1.5) and (1.13) we get (1.14) $Y \subseteq Z$. On the other hand, from (1.4), (1.14) and the fact that cons is hereditary we conclude that (1.15) $Z \notin$ cons contrary to (1.10). This proves step (1). Now let (2) $A \notin$ cn[cons](Y). By (2) and the definition of cn[cons] we have step (3)

$Y \cup \{\neg A\} \in$ cons. By (1) and (3) we get step (4) cn[cons]$(Y, \neg A) \in$ cons. Given that it has already been shown in the course of this proof that cn[cons] is both reflexive and monotonic we can now conclude that (5) cn[cons]$(Y) \cup \{\neg A\} \subseteq$ cn[cons]$(Y, \neg A)$. From (4) and (5) we infer step (6) cn[cons]$(Y) \cup \{\neg A\} \in$ cons, because cons is hereditary. By another use of the definition of cn[cons] we conclude from (5) that $A \notin$ cn[cons](cn[cons](Y)). This shows that cn[cons] is idempotent.

Proof that cn[cons] *is finitary*: Suppose, to the contrary, that (1) $A \in$ cn[cons](X) and (2) $A \notin$ cn[cons](Y) for any $Y \in \mathrm{fin}(X)$. From (1) and the definition of cn[cons] we get (3) $X \cup \{\neg A\} \notin$ cons. By (3) there is Z such that (4) $Z \in \mathrm{fin}(X, \neg A)$ and (5) $Z \notin$ cons because cons is compact. Step (4) implies that (6) Z-$\{\neg A\} \in \mathrm{fin}(X)$. From (2) and (6) we can infer that (7) $A \notin$ cn[cons]$(Z$-$\{\neg A\})$. And from (7) and the definition of cn[cons] we can get that (8) $Z \cup \{\neg A\} \in$ cons. But cons is hereditary. Hence, by (8), $Z \in$ cons contrary to (5).

Proof that cn[cons] *is* $(\neg)$-*analytic*: If we suppose, to the contrary, that cn[cons]$(A, \neg A) \neq S$ then there is $B \notin S$ such that $B \notin$ cn[cons]$(A, \neg A)$ and then, by the definition of cn[cons], $\{A, \neg A, \neg B\} \in$ cons and this implies that cons is not $(\neg)$-analytic contrary to the hypothesis.

Proof that cn[cons] *is* $(\neg)$-*synthetic*: Suppose, to the contrary, that (1) $B \in$ cn[cons](X, A), (2) $B \in$ cn[cons]$(X, \neg A)$ and (3) $B \notin$ cn[cons](X). Step (3) and the definition of cn[cons] imply that (4) $X \cup \{\neg B\} \in$ cons. By the hypothesis cons is $(\neg)$-synthetic. Hence by (4) we have step (5) $X \cup \{A, \neg B\} \in$ cons or $X \cup \{\neg A, \neg B\} \in$ cons. From (2) and the definition of cn[cons] we conclude that (6) $X \cup \{\neg A, \neg B\} \notin$ cons while from (5) and (6) we get step (7) $X \cup \{A, \neg B\} \in$ cons. By another use of the definition of cn[cons] and step (7) we infer that $B \notin$ cn(X, A) contrary to (1).

(ii) *Proof that* cons[cn[cons]] $\subseteq$ cons: If $X \in$ cons[cn[cons]] then, by the definition of cons[cn[cons]] in terms of cn[cons], cn[cons]$(X) \neq S$, i.e. $A \notin$ cn[cons](X) for some A. Hence, by the definition of cn[cons], $X \cup \{\neg A\} \in$ cons, i.e. $X \in$ cons because cons is hereditary.

Proof that cons $\subseteq$ cons[cn[cons]]: Suppose, to the contrary, that (1) $X \in$ cons and (2) $X \notin$ cons[cn[cons]]. By the hypothesis cons is a compact $(\neg)$-consistency property so, by Lemma 5, it is regular. In other words, step (1) implies that there is Y such that (3) $Y \in 2_X$, (4) $Y \in$ cons and (5) $Z = Y$ for any $Z \in$ cons $\cap\, 2_Y$. To continue this proof we will now prove step (6) cn[Cons]$(X) \subseteq Y$. To do so suppose that (6.1) $A \in$ cn[cons](X) and (6.2) $A \notin Y$. By (6.1) and the definition of cn[cons], $X \cup \{\neg A\} \notin$ cons. Using (3) we can conclude here that (6.3) $Y \cup \{\neg A\} \notin$ cons because cons is hereditary. From (4) and (6.3) we infer (6.4) $Y \cup \{A\} \in$ cons because cons is $(\neg)$-synthetic. From (5) and (6.4) it follows, finally, that $A \in Y$ contrary to (6.2). This proves step (6). By (2) and the definition of cons[cn[cons]] in terms of cn[cons] we also conclude that (7) cn[cons]$(X) = S$. Steps (6) and (7) imply that $Y = S$, i.e. that (8) $Y \notin$ cons because cons is non-trivial. But step (8) is contrary to (4). $\square$

By Lemmas 6 and 7 the pair (cons[cn], cn[cons]) is a connection between finitary $(\neg)$-closure operators and compact $(\neg)$-consistency properties. What remains to be proved yet is that the connection is antitone and this is provided by our next lemma.

Lemma 8. (i) *If* $\mathrm{cn}_1 \subseteq \mathrm{cn}_2$ *then* $\mathrm{cons}[\mathrm{cn}_2] \subseteq \mathrm{cons}[\mathrm{cn}_1]$ *for any finitary* $(\neg)$*-closure operators* cn_1 *and* cn_2 *over* S. (ii) *If* $\mathrm{cons}_1 \subseteq \mathrm{cons}_2$ *then* $\mathrm{cn}[\mathrm{cons}_2] \subseteq \mathrm{cn}[\mathrm{cons}_1]$ *for any compact* $(\neg)$*-consistency properties* cons_1 *and* cons_2 *over* S.

Proof. (i) Suppose that (1) $\mathrm{cn}_1 \subseteq \mathrm{cn}_2$. If $X \in \mathrm{cons}[\mathrm{cn}_2]$ then, by the definition of $\mathrm{cons}[\mathrm{cn}_2], \mathrm{cn}_2(X) \neq S$ then, by (1), $\mathrm{cn}_1(X) \neq S$ and, by the definition of $\mathrm{cons}[\mathrm{cn}_1], X \in \mathrm{cons}[\mathrm{cn}_1]$ so $\mathrm{cons}[\mathrm{cn}]$ is antitone.

(ii) Let (1) $\mathrm{cons}_1 \subseteq \mathrm{cons}_2$. If $A \in \mathrm{cn}[\mathrm{cons}_2](X)$ then, by the definition of $\mathrm{cn}[\mathrm{cons}_2]$, $X \cup \{\neg A\} \notin \mathrm{cons}_2$ then, by (1), $X \cup \{\neg A\} \notin \mathrm{cons}_1$, by the definition of $\mathrm{cons}[\mathrm{cn}_1]$, $A \in \mathrm{cn}[\mathrm{cons}_1](X)$. □

Lemmas 6–8 justify the following theorem.

Theorem 4. *The pair* (cons[cn], cn[cons]) *is a Galois* (*or antitone*) *connection between finitary* $(\neg)$*-closure operators and compact* $(\neg)$*-consistency properties over* S.

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